Systems for Sparse Representations: Fourier Analysis, Wavelets & Shearlets

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DEDALE Tutorial Day
November 16, 2016

Applied Functional Analysis Group
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Outline

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1. Wavelets

2. Curvelets & Shearlets

Definition: Let $\phi, \psi \in L^2(\mathbb{R})$ be a scaling function and a wavelet (i.e., $\int_{\mathbb{R}} |\widehat{\psi}(\omega)|^2 \frac{d\omega}{|\omega|} < \infty$). Then the **inhomogeneous discrete** wavelet system generated by ϕ, ψ is

$$\Big(\phi(\bullet - m)\Big)_{m\in\mathbb{Z}} \cup \Big(2^{j/2} \cdot \psi(2^j \bullet - m)\Big)_{j\in\mathbb{N}_0, m\in\mathbb{Z}}.$$

Wavelets in 1D and 2D

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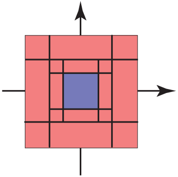
Definition: Let ϕ, ψ as above. A 2D inhomogeneous discrete wavelet system is defined by

$$\Big(\phi^{(1)}(\bullet - m)\Big)_{m\in\mathbb{Z}^2} \cup \Big(2^{j} \cdot \psi^{(\ell)}(2^j \bullet - m)\Big)_{j\in\mathbb{N}_0, m\in\mathbb{Z}^2, \ell\in\{1,2,3\}}$$



where

$$\phi^{(1)} := \phi \otimes \phi, \quad \psi^{(1)} := \phi \otimes \psi, \quad \psi^{(2)} := \psi \otimes \phi, \quad \psi^{(3)} := \psi \otimes \psi.$$

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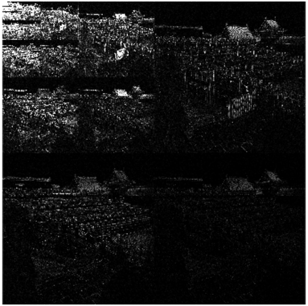

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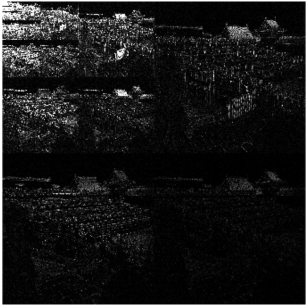
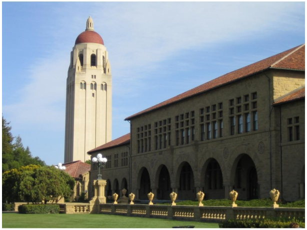


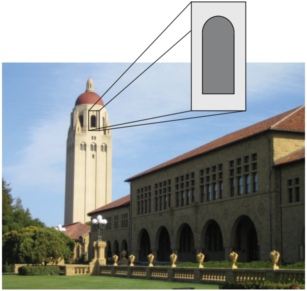
Theorem: Discrete wavelets provide optimal approximation rates for those $f \in L^2(\mathbb{R}^2)$ which are C^2 apart from **point singularities**:

$$\|f - f_N\|_{L^2} \lesssim N^{-1/2} \qquad (N \to \infty).$$

Living on the point



Living on the ~~point~~ edge



Natural images are governed by curved singularities, not point singularities!

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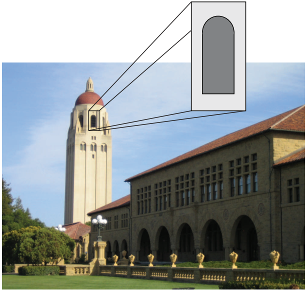

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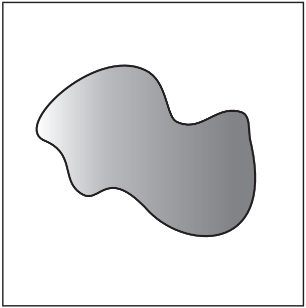

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Definition (Donoho; 2001)

With $Q := (0,1)^2$, the set of cartoon-like functions is defined as

$$\mathcal{E}^2(\mathbb{R}^2) = \left\{ f_0 + f_1 \cdot \mathbb{1}_B \,\middle|\, \partial B \subset Q \text{ closed } C^2 \text{ curve and } f_0, f_1 \in C_c^2(Q) \right\}.$$

Approximation of cartoon-like functions

Theorem: For $f \in \mathcal{E}^2\left(\mathbb{R}^2\right)$, the best N-term approximation f_N

- using Fourier basis: $\|f - f_N\|_{L^2} \lesssim N^{-1/4}$,
- using Wavelets: $\|f - f_N\|_{L^2} \lesssim N^{-1/2}$.

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Theorem (Donoho; 2001): Let $(\psi_n)_{n \in \mathbb{N}}$ be **any** countable family in $L^2(\mathbb{R}^2)$. If

$$\|f - f_N\|_{L^2} \lesssim N^{-\theta} \qquad \forall N \in \mathbb{N} \qquad \forall f \in \mathcal{E}^2(\mathbb{R}^2),$$

then $\theta \leq 1$. Here, we assume polynomial depth search for forming f_N.

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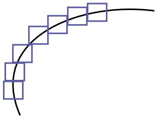
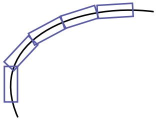
Intuitive explanation: This is caused by the scalar dilations:

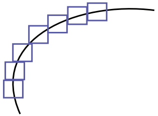


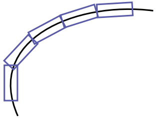


Outline

Towards a directional system

Recall: Isotropic dilations are suboptimal for approximating curves:

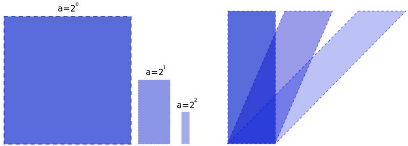




Recall: Isotropic dilations are suboptimal for approximating curves:

Idea for shearlets: Use

- Parabolic scaling
- Different orientations via shearing.



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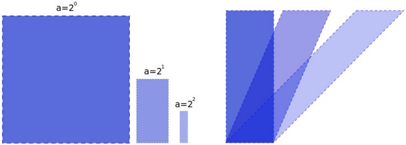

Advantages of shearing:

- The shearing matrices $\left(\begin{smallmatrix}1 & k\\ 0 & 1\end{smallmatrix}\right)$ leave the digital grid $\mathbb{Z}^2$ invariant.
- Uniform theory for the analog and digital situation.

Definition (Kutyniok, Labate; 2006): For $\psi \in L^2\left(\mathbb{R}^2\right)$, the associated discrete shearlet system is

$$\left(2^{\frac{3}{4}j} \cdot \psi\left(S_k A_{2^j} \bullet - m\right)\right)_{j,k\in\mathbb{Z},m\in\mathbb{Z}^2}.$$

A first definition of shearlets

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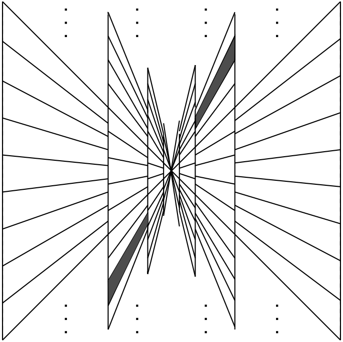
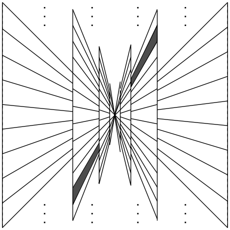
Induced frequency tiling:



The induced frequency tiling



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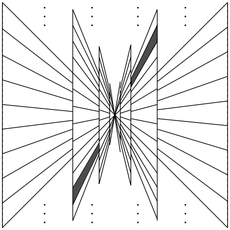

Very different treatment of x-direction and y-direction!

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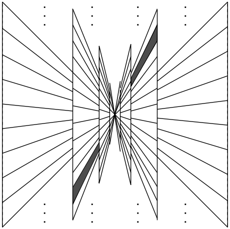

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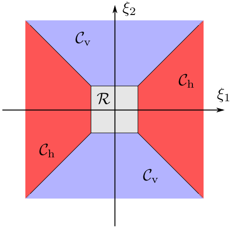
Solution: Use cone-adapted shearlets:



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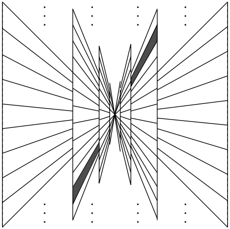

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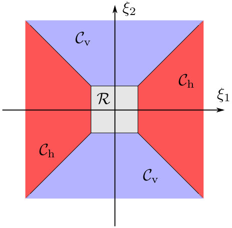

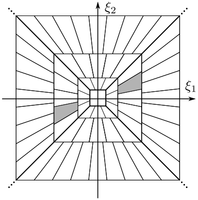

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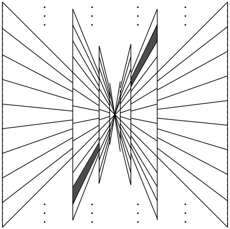

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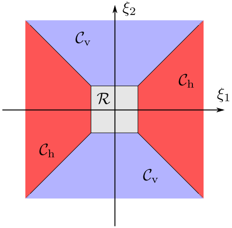
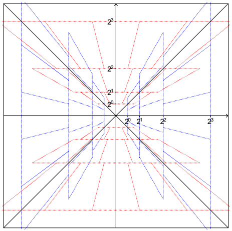




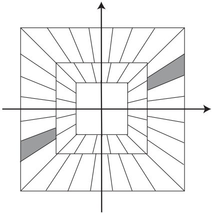
Cone-adapted Discrete Shearlet Systems

Definition (Kutyniok, Labate; 2006):
The cone-adapted discrete shearlet system $\mathcal{SH}(\phi, \psi, \tilde{\psi}; c)$ with sampling density $c > 0$ generated by $\phi, \psi, \tilde{\psi} \in L^2(\mathbb{R}^2)$ is the union of

$$\left\{\phi(\bullet - c \cdot m) \,\middle|\, m \in \mathbb{Z}^2\right\},$$
$$\left\{\psi_{j,k,m,h} := \psi(S_k A_{2^j} \bullet - c \cdot m) \,\middle|\, j \in \mathbb{N}_0, |k| \leq \lceil 2^{j/2} \rceil, m \in \mathbb{Z}^2\right\},$$
$$\left\{\psi_{j,k,m,v} := \tilde{\psi}\left(S_k^T \tilde{A}_{2^j} \bullet - c \cdot m\right) \,\middle|\, j \in \mathbb{N}_0, |k| \leq \lceil 2^{j/2} \rceil, m \in \mathbb{Z}^2\right\},$$

where

$$S_k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, \qquad A_{2^j} = \begin{pmatrix} 2^j & 0 \\ 0 & 2^{j/2} \end{pmatrix}, \qquad \tilde{A}_{2^j} = \begin{pmatrix} 2^{j/2} & 0 \\ 0 & 2^j \end{pmatrix}.$$



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Side remark: Curvelets

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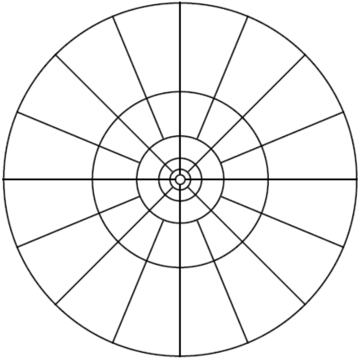

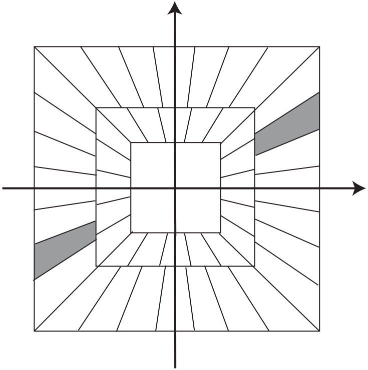

Theorem (Kittipoom, Kutyniok, Lim; 2012):
Let $\phi, \psi, \tilde{\psi} \in L^2\left(\mathbb{R}^2\right)$ be compactly supported and assume that

- $\widehat{\psi}, \widehat{\tilde{\psi}}$ satisfy certain decay conditions,
- we have

$$|\widehat{\phi}(\xi)|^2 + \sum_{j,k} |\widehat{\psi}_{j,k}(\xi)|^2 + |\widehat{\tilde{\psi}_{j,k}}(\xi)|^2 \geq C > 0 \quad \text{a.e.} \qquad (\dagger)$$

Then there is some $c > 0$ such that $\mathcal{SH}(\phi, \psi, \tilde{\psi}; c)$ is a frame for $L^2\left(\mathbb{R}^2\right)$.

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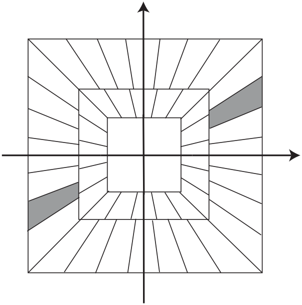

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Remarks

- For $(\dagger)$, it suffices to have $\tilde{\psi}((\xi_1, \xi_2)) = \psi((\xi_2, \xi_1))$ as well as

$$\begin{aligned} \left|\widehat{\phi}(\xi)\right| &\gtrsim 1 && \text{for } \xi \in [-1,1]^2, \\ |\widehat{\psi}(\xi)| &\gtrsim 1 && \text{for } \xi_1 \in [1/3,\, 3] \text{ and } |\xi_2| \leq |\xi_1|, \end{aligned}$$

- There are special examples with frame bounds $B/A \approx 4$.

Theorem (Kutyniok, Lim; 2011):
Let $\phi, \psi, \tilde{\psi} \in L^2\left(\mathbb{R}^2\right)$ be compactly supported and such that $\mathcal{SH}(\phi, \psi, \tilde{\psi}; c)$ is a frame for $L^2\left(\mathbb{R}^2\right)$. Further, assume that $\widehat{\psi}, \widehat{\tilde{\psi}}$ satisfy certain decay conditions. Then $\mathcal{SH}(\phi, \psi, \tilde{\psi}; c)$ provides an optimally sparse approximation of all $f \in \mathcal{E}^2\left(\mathbb{R}^2\right)$, i.e.,

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Remark

The proof shows

$$\sum_{n>N} |\theta(f)|_n^2 \lesssim N^{-2} \cdot (\log N)^3 \qquad \forall N \in \mathbb{N} \qquad \forall f \in \mathcal{E}^2(\mathbb{R}^2),$$

where $|\theta(f)|_n$ denotes the n-th largest shearlet coefficient. Hence, the shearlet coefficients are in ℓ^p for all $p > \frac{2}{3}$.

Thank you!

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Questions, comments, counterexamples?

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